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Oscillating Decay of an Unstable System

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We study the medium-time behavior of the survival probability in the frame of the N-level Friedrichs model. The time evolution of an arbitrary unstable initial state is determined. We show that the survival probability may oscillate significantly during the so-called exponential era. This result explains qualitatively the experimental observations of the NaI decay. The Gamow states for N-level Friedrichs model are constructed. The time evolution in terms of the complex spectral representation including the Gamow states is discussed.

KEY WORDS: unstable system; non-exponential decay; resonances; Friedrichs model.

1. INTRODUCTION

Recent developments in femtosecond laser optics (see for example paper (Zewail, 2000) and the XXth Solvay Conference on Chemistry (Gaspard and Burghardt, 1997)) opened new possibilities for the study of quantum transitions, which are a very important subject of the quantum theory. In a series of works, Zewail *et al.* (Cong *et al.*, 1996; Felker and Zewail, 1995; Lienau and Zewail, 1996; Potter *et al.*, 1992) applied femtosecond transition-time spectroscopy for the probing of chemical reactions. Following the work of Kinsey *et al.* (Imre *et al.*, 1984) they attempted in paper (Cong *et al.*, 1996) to track wave packet trajectories in the dissociation of NaI.

The shapes of the ground state potential for NaI and of the quasi-bound potential of the $Na(^2S_{1/2}) + I(^2P_{3/2})$ system suggest a mechanism of the induced dissociation process. The femtosecond laser pulse brings the NaI molecule to the state of quasi-bound ions. The distance between the ions reaches the region where

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two potentials have similar values due to vibrations of a NaI excited state. Then the transition from $Na(^2S_{1/2}) + I(^2P_{3/2})$ quasi-bound state to NaI continuum state occurs resulting in the dissociation of the molecule.

After an initial exciting laser pulse, the experiment shows oscillations of the $Na(^2S_{1/2}) + I(^2P_{3/2})$ population, which are explained in (Cong *et al.*, 1996) by wave packet propagation. The direction of the wave packet propagation is correlated with the oscillation (extension and contraction) of the NaI bond. The quantum dynamics calculations are based on a time-dependent perturbation formalism.

This problem is an example of the interaction of the discrete spectrum with the continuous spectrum, which was extensively discussed in the literature starting from the work of Friedrichs (1948). Indeed, the energy states of $Na(^2S_{1/2}) + I(^2P_{3/2})$ are the excited state embedded into the continuum states of the decay products. Therefore, the time-dependence of the $Na(^2S_{1/2}) + I(^2P_{3/2})$ population is described by the survival probability of the excited state prepared by the laser pulse.

The original Friedrichs model (Friedrichs, 1948) contains two discrete energy levels a ground state and an excited state, coupled with the continuum, being bounded from below. The time-dependence of the survival probability of the excited state has been studied both theoretically (Antoniou *et al.*, 2001; Facchi *et al.*, 2001; Facchi and Pascazio, 1999; Khalfin, 1957, 1958; Kofman *et al.*, 1994; Kofman and Kurizki, 2000; Namiki *et al.*, 1997) and experimentally (Balzer *et al.*, 2000; Fischer *et al.*, 2001; Itano *et al.*, 1990). It is exponential with a short nonexponential initial era and a nonexponential long tail. As a result, Friedrichs models are very appropriate for the discussion of the particle decay and for the description of dressed unstable states (Antoniou and Prigogine, 1993; Ordonez, *et al.*, 2001; Petrosky *et al.*, 1991). The analytical structure of the N-level Friedrichs model has been analyzed (Bayley and Schieve, 1978; Davies, 1974; Duerinckx, 1983; Exner, 1985; Ruuskanen, 1970; Stey and Gibberd, 1972), and the oscillations of the survival probability were discussed, for example, in papers (Alicki and Lendi, 1987; Hegerfeldt and Plenio, 1992, 1993; Kofman, *et al.*, 1994; Lendi, 1980; Ruuskanen, 1970).

In the present paper we shall show that the N-level Friedrichs model can also explain the oscillations of the survival probability of the excited state observed by Zewail and coworkers (Cong *et al.*, 1996). Several excited levels are necessary in order to construct a wave packet, which can exhibit localization and nonconventional time evolution. In Section 2 we present the model and describe the exact solution diagonalizing the Hamiltonian. Using the relation between eigenstates of the unperturbed Hamiltonian and the total Hamiltonian, we describe in Section 3 the time evolution of the basis states. Specifying the form factor of the interaction, we show in Section 4 the appearance of oscillations already for the two level Friedrichs model. In Section 5 we demonstrate that the survival probability of unstable states in the N-level Friedrichs model is in fact very close to the one obtained in the experiment (Cong *et al.*, 1996).

2. MODEL AND EXACT SOLUTION

The Hamiltonian of the Friedrichs model (Friedrichs, 1948) generalized to N-level is

$$H = H_0 + \lambda V, \tag{1}$$

where

$$H_0 = \sum_{k=1}^{N} \omega_k |k\rangle\langle k| + \int_0^\infty d\omega\ \omega|\omega\rangle\langle\omega|,$$

$$V = \sum_{k=1}^{N} \int_0^\infty d\omega\ \ f_k(\omega)\,(|k\rangle\langle\omega| + |\omega\rangle\langle k|). \tag{2}$$

Here $|k\rangle$ represent states of the discrete spectrum with the energy ω_k, $\omega_k > 0$. We assume the simplest case that $\omega_k \neq \omega'_k$ for $k \neq k'$. The vectors $|\omega\rangle$ represent states of the continuous spectrum with the energy ω, $f_k(\omega)$ are the form factors for the transitions between the discrete and the continuous spectrum, and λ is the coupling parameter. The vacuum energy is chosen to be zero. The states $|k\rangle$ and $|\omega\rangle$ form a complete orthonormal basis:

$$\langle k|k'\rangle = \delta_{kk'}, \qquad \langle\omega|\omega'\rangle = \delta(\omega - \omega'), \qquad \langle\omega|k\rangle = 0, \quad k, k' = 1 \ldots N, \tag{3}$$

$$\sum_{k=1}^{N} |k\rangle\langle k| + \int_0^\infty d\omega\ |\omega\rangle\langle\omega| = I, \tag{4}$$

where $\delta_{kk'}$ is the Kronecker symbol, $\delta(\omega - \omega')$ is the Dirac delta function and I is the unity operator. The Hamiltonian H_0 has the continuous spectrum on the interval $[0, \infty)$ and the discrete spectrum $\omega, \ldots, \omega_k$ embedded in the continuous spectrum.

As the interaction λV is switched on, the eigenstates $|k\rangle$ become resonances of H as in the case of the one-level Friedrichs model (Friedrichs, 1948). Let us consider the eigenvalue problem for the N-level Friedrichs Hamiltonian (1)

$$H|\Psi_\omega\rangle = \omega|\Psi_\omega\rangle. \tag{5}$$

We shall look for the solution of Eq. (5) in the form:

$$|\Psi_\omega\rangle = \sum_k \psi_k(\omega)|k\rangle + \int_0^\infty d\omega' \psi(\omega, \omega')|\omega'\rangle, \tag{6}$$

where $\psi_k(\omega)$ and $\psi(\omega, \omega')$ are unknown functions. Inserting (6) into (5) and making use of the orthogonality relations, we obtain for them a system of equations:

$$\begin{cases} (\omega_k - \omega)\psi_k(\omega) + \lambda \int_0^\infty d\omega'\, f_k(\omega')\psi(\omega, \omega') = 0, \\ (\omega' - \omega)\psi(\omega, \omega') + \lambda \sum_{k=1}^{N} f_k(\omega')\psi_k(\omega) \qquad = 0. \end{cases} \tag{7}$$

Eliminating $\psi(\omega, \omega')$ from this system, we arrive at the following equation for $\psi_k(\omega)$:

$$\sum_{k'=1}^{N} G_{kk'}^{-1}(\omega)\psi_{k'}(\omega) = -C\lambda f_k(\omega), \tag{8}$$

where C is an arbitrary constant. $G_{kk'}(\omega)$ are the matrix elements of the partial resolvent which is

$$G_{kk'}^{-1}(\omega) = (\omega_k - \omega)\delta_{kk'} - \lambda^2 \int_0^\infty d\omega' \frac{f_k(\omega') f_{k'}(\omega')}{\omega' - \omega}. \tag{9}$$

Under certain conditions (which will be specified below, see also (Exner, 1985)), the function $G_{kk'}(z)$ is analytic everywhere in the first sheet of the Riemann manifold except for the cut $[0, \infty)$. In this case, the Hamiltonian H has no discrete spectrum. The solution of Eq. (8) is given by

$$\psi_k(\omega) = -C\lambda \sum_{k'=1}^{N} G_{kk'}(\omega \pm i0) f_{k'}(\omega). \tag{10}$$

With this equation we find $\psi(\omega, \omega')$ from the system (7):

$$\psi(\omega, \omega') = C\left[\delta(\omega - \omega') + \frac{\lambda^2 \sum_{k,k'=1}^{N} f_k(\omega') G_{kk'}(\omega \pm i0) f_{k'}(\omega)}{\omega - \omega' \pm i0}\right]. \tag{11}$$

The eigenvalue problem (5) has two sets of solutions

$$|\Psi_\omega\rangle_{\substack{\text{in}\\ \text{out}}} = |\omega\rangle + \lambda \sum_{k,l=1}^{N} f_l(\omega) G_{kl}(\omega \pm i0) \left\{\int_0^\infty \mathrm{d}\omega' \frac{\lambda f_k(\omega')}{\omega' - \omega \mp i0}|\omega'\rangle - |k\rangle\right\}, \tag{12}$$

which correspond to the "in" and "out" asymptotic conditions. The value $C = 1$ corresponds to the orthonormalization condition

$$_{\substack{\text{in}\\ \text{out}}}\langle\Psi_\omega|\Psi_{\omega'}\rangle_{\substack{\text{in}\\ \text{out}}} = \delta(\omega - \omega'). \tag{13}$$

We can also prove the completeness condition

$$\int_0^\infty \mathrm{d}\omega |\Psi_\omega\rangle_{\substack{\text{in}\\ \text{out}}} {}_{\substack{\text{in}\\ \text{out}}}\langle\Psi_\omega| = \sum_{k=1}^{N} \omega_k |k\rangle\langle k| + \int_0^\infty d\omega |\omega\rangle\langle\omega|. \tag{14}$$

Hence the new states diagonalize the total Hamiltonian (1) as

$$H = \int_0^\infty \mathrm{d}\omega\, \omega |\Psi_\omega\rangle_{\substack{\text{in}\\ \text{out}}} {}_{\substack{\text{in}\\ \text{out}}}\langle\Psi_\omega|. \tag{15}$$

The proof of completeness is based on the matrix formula

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1},$$

from which we can derive:

$$G_{kk'}(\omega + i0) - G_{kk'}(\omega - i0) = 2\pi i \lambda^2 \sum_{l,m=1}^{N} G_{kl}(\omega + i0) f_l(\omega) f_m(\omega) G_{mk'}(\omega - i0). \tag{16}$$

Using the asymptotics:

$$G_{kk'}(\omega) \underset{\omega\to\infty}{\longrightarrow} \frac{\delta_{kk'}}{\omega - \omega'} + O\left(\frac{1}{\omega - \omega'}\right), \tag{17}$$

we prove other useful relations for G:

$$G_{kk'}(\omega \pm i0) = \lambda^2 \int_0^\infty d\omega' \sum_{l,m=1}^{N} f_l(\omega') f_m(\omega') \frac{G_{kl}(\omega' + i0) G_{mk'}(\omega' - i0)}{\omega' - \omega \mp i0}, \tag{18}$$

and

$$\lambda^2 \int_0^\infty d\omega \sum_{l,m=1}^{N} f_l(\omega) f_m(\omega) G_{kl}(\omega + i0) G_{mk'}(\omega - i0) = \delta_{kk'}. \tag{19}$$

Because of the completeness of the new basis (14) the old basis vectors may be expressed in terms of the new ones as

$$|k\rangle = \int_0^\infty d\omega\, |\Psi_\omega\rangle_{\rm in}\, {}_{\rm in}\langle \Psi_\omega | k\rangle, \qquad |\omega\rangle = \int_0^\infty d\omega' |\Psi_{\omega'}\rangle_{\rm in}\, {}_{\rm in}\langle \Psi_{\omega'} | \omega\rangle \tag{20}$$

where ${}_{\rm in}\langle \Psi_\omega | k\rangle$ and ${}_{\rm in}\langle \Psi_{\omega'} | \omega\rangle$ are the complex conjugates of $\langle k|\Psi_\omega\rangle_{\rm in}$ and $\langle \omega|\Psi_{\omega'}\rangle_{\rm in}$ respectively, which may be obtained from (12):

$$\langle k|\Psi_\omega\rangle_{\rm in} = -\lambda \sum_{l=1}^{N} f_l(\omega) G_{kl}(\omega + i0), \tag{21}$$

$$\langle \omega|\Psi_{\omega'}\rangle_{\rm in} = \delta(\omega - \omega') - \sum_{k,l=1}^{N} \frac{\lambda^2 f_k(\omega) f_l(\omega') G_{k,l}(\omega')}{\omega' - \omega - i0}. \tag{22}$$

Inserting complex conjugate of (21) into (20) we obtain the inverse relations in the form:

$$|k\rangle = -\lambda \sum_{l=1}^{N} \int_0^\infty d\omega\;\; f_l(\omega) G_{kl}(\omega - i0) |\Psi_\omega\rangle_{\rm in} \tag{23}$$

$$|\omega\rangle = |\Psi_\omega\rangle_{\rm in} - \sum_{k,l=1}^{N} \lambda f_k(\omega) \int_0^\infty d\omega' \frac{\lambda f_l(\omega') G_{k,l}(\omega')}{\omega' - \omega - i0} |\Psi_{\omega'}\rangle_{\rm in}. \tag{24}$$

These inverse relations will be used for the calculation of the time evolution of $|k\rangle$ and $|\omega\rangle$ in the next section.

3. TIME EVOLUTION

Using the known evolution of the state $|\Psi_\omega\rangle_{\rm in}$,

$$e^{-iHt}|\Psi_\omega\rangle_{\rm in} = e^{-i\omega t}|\Psi_\omega\rangle_{\rm in},$$

we can find the evolution of the eigenstates of H_0:

$$|k\rangle_t = -\lambda \sum_{l=1}^{N} \int_0^\infty d\omega\, e^{-i\omega t} f_l(\omega) G_{kl}(\omega - i0)|\Psi_\omega\rangle_{\rm in}, \tag{25}$$

$$\begin{aligned}|\omega\rangle_t &= e^{-i\omega t}|\Psi_\omega\rangle_{\rm in}\\ &\quad - \sum_{k,l=1}^{N} \lambda f_k(\omega) \int_0^\infty d\omega'\, e^{-i\omega'} \frac{\lambda f_l(\omega') G_{kl}(\omega')}{\omega' - \omega - i0}|\Psi_{\omega'}\rangle_{\rm in}.\end{aligned} \tag{26}$$

Using (12), we obtain the representation

$$|k\rangle_t = \sum_{l=1}^{N} A_{kl}(t)|l\rangle + \lambda \sum_{l=1}^{N} \int_0^\infty d\omega\; f_l(\omega) g_{kl}(\omega, t)|\omega\rangle, \tag{27}$$

$$\begin{aligned}|\omega\rangle_t &= e^{-i\omega t}|\omega\rangle - \lambda^2 \sum_{k,l=1}^{N} f_l(\omega) \int_0^\infty d\omega'\; f_k(\omega) \frac{g_{kl}(\omega', t) - g_{kl}(\omega, t)}{\omega' - \omega}\\ &\quad + \sum_{k,l=1}^{N} \lambda f_k(\omega) g_{kl}(\omega, t)|l\rangle\end{aligned} \tag{28}$$

in terms of the time-dependent matrix functions $A_{kl}(t)$ and $g(\omega, t)$:

$$A_{kl}(t) = \lambda^2 \sum_{l,m,n=1}^{N} \int_0^\infty d\omega\, e^{-i\omega t} f_m(\omega) f_n(\omega) G_{km}(\omega + i0) G_{ln}(\omega - i0), \tag{29}$$

$$\begin{aligned}g_{kl}(\omega, t) &= -e^{-i\omega t} G_{kl}(\omega - i0)\\ &\quad + \lambda^2 \sum_{m,n=1}^{N} \int_0^\infty d\omega'\, e^{-i\omega' t} \frac{f_m(\omega') f_n(\omega') G_{km}(\omega' - i0) G_{ln}(\omega' + i0)}{\omega' - \omega + i0}.\end{aligned} \tag{30}$$

With the help of (16), we can rewrite (29) in the form

$$\begin{aligned}A_{kl}(t) &= \frac{1}{2\pi i} \int_0^\infty d\omega\, e^{-i\omega t}(G_{kl}(\omega + i0) - G_{kl}(\omega - i0))\\ &= \frac{1}{2\pi i} \int_{\rm C} {\rm d}\omega\, e^{-i\omega t} G_{kl}(\omega),\end{aligned} \tag{31}$$

where the contour C is shown in Fig. 1.

With the help of (18), we rewrite (30) in the form

$$g_{kl}(\omega, t) = \lambda^2 \sum_{m,n=1}^{N} \int_0^\infty d\omega' \, f_m(\omega') f_n(\omega') G_{km}(\omega' - i0) \times G_{ln}(\omega' + i0) \frac{e^{-i\omega' t} - e^{-i\omega t}}{\omega' - \omega + i0}. \tag{32}$$

The integrand in (32) does not have any singularity at $\omega' = \omega$, therefore $i0$ in the denominator becomes redundant. Then using (16) we obtain

$$\begin{aligned} g_{kl}(\omega, t) &= \frac{1}{2\pi i} \int_0^\infty d\omega' \, (G_{kl}(\omega' + i0) - G_{kl}(\omega' - i0)) \frac{e^{-i\omega' t} - e^{-i\omega t}}{\omega' - \omega} \\ &= \frac{1}{2\pi i} \int_{\mathrm{C}} \mathrm{d}\omega' G_{kl}(\omega') \frac{e^{-i\omega' t} - e^{-i\omega t}}{\omega' - \omega}, \end{aligned} \tag{33}$$

where the contour C is shown in Fig. 1. For real $\omega > 0$ the term with the factor $e^{-i\omega t}$ vanishes because it does not have any singularities outside the positive part of the real line. Then we have

$$g_{kl}(\omega, t) = \frac{1}{2\pi i} \int_{\mathrm{C}} \mathrm{d}\omega' G_{kl}(\omega') \frac{e^{-i\omega' t}}{\omega' - \omega}. \tag{34}$$

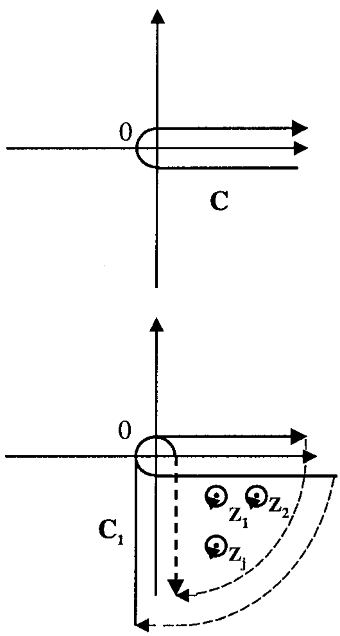

Fig. 1. The contours of integration C and C_1.

One can easily check the following relation between $A_{kl}(t)$ and $g_{kl}(\omega, t)$:

$$A_{kl}(t) = \left(i \frac{\mathrm{d}}{\mathrm{d}t} - \omega \right) g_{kl}(\omega, t). \tag{35}$$

The time evolution of a state $|\Phi\rangle$, which is a superposition of the eigenstates of H_0,

$$|\Phi\rangle = \sum_{k=1}^{N} a_k |k\rangle, \tag{36}$$

may be obtained with the help of (27)

$$|\Phi(t)\rangle = \sum_{k=1}^{N} a_k |k\rangle_t. \tag{37}$$

The survival amplitude $A(t)$ of this state is

$$A(t) \equiv \langle \Phi | \Phi(t) \rangle = \sum_{k,k'=1}^{N} a_k a_{k'}^{*} \langle k | k' \rangle_t = \sum_{k,k'=1}^{N} a_k a_{k'}^{*} A_{kk'}(t). \tag{38}$$

Changing the contour of the integration C to C_1 in $A_{kk'}(t)$ as shown in Fig. 1, we arrive at

$$A_{kk'}(t) = -\sum_{j} r_{kk'}^{j} e^{-iz_j t} + \frac{1}{2\pi i} \int_{\mathrm{C}_1} \mathrm{d}\omega \, e^{-i\omega t} G_{kk'}(\omega), \tag{39}$$

where $r_{kk'}^{j}$ is the residue of $G_{kk'}(\omega)$ at the pole z_j:

$$r_{kk'}^{j} = \operatorname{res} G_{kk'}(\omega)|_{\omega = z_j}. \tag{40}$$

The first term in (39) corresponds to the contribution of the poles z_j while the second term is the background integral, which gives rise to so-called long tail behavior (Facchi and Pascazio, 1999; Khalfin, 1957, 1958). It is known that the integral term plays essential role for very long as well as very short times. In the case of very short times we have the well-known Zeno and anti-Zeno regions (Antoniou *et al.*, 2001; Fischer *et al.*, 2001; Kofman and Kurizki, 2000; Namiki *et al.*, 1997). If we consider the intermediate "exponential decay" era, the integral term can be neglected because in this time scale, it is of the next order in λ^2 compared with the first term.

The same result for $A_{kk'}(t)$ (39) is obtained in Appendix in terms of Gamow vectors (A14). In the intermediate "exponential" era, the main contribution to the survival probability comes from the Gamow vectors as one may neglect the integral term arising from the background.

4. TWO LEVEL MODEL

The rich structure of the model involving more than one level, will be first illustrated with example with two excited levels by choosing the form factor in the form similar to (Likhoded and Pronko, 1997)

$$f_k(\omega) = \frac{\omega^{1/4}}{\omega + \rho_k^2}. \tag{41}$$

For this form factor the matrix element $G^{-1}_{kk'}(\omega)$ (9) is

$$G^{-1}_{kk'}(\omega) = (\omega_k - \omega)\delta_{kk'} + \frac{\pi\lambda^2}{\rho_k + \rho_{k'}} \frac{1}{(\sqrt{\omega} + i\rho_k)\,(\sqrt{\omega} + i\rho_{k'})}, \tag{42}$$

where the first sheet of the complex ω plane corresponds to the upper half of the complex $\sqrt{\omega}$ plane. The square root is defined with the cut $[0, +\infty)$ such that $\sqrt{\omega} > 0$ at the upper rim of the cut. For $\rho_k > 0$ all singularities of the integral in expression (9) are on the second sheet. In the case of two levels the matrix is

$$G^{-1}(\omega) = \begin{pmatrix} (\omega_1 - \omega) + \dfrac{\pi\lambda^2}{2\rho_1(\sqrt{\omega} + i\rho_1)^2} & \dfrac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} \\ \dfrac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} & (\omega_2 - \omega) + \dfrac{\pi\lambda^2}{2\rho_2(\sqrt{\omega} + i\rho_2)^2} \end{pmatrix} \tag{43}$$

The 2×2 matrix representing the partial resolvent is

$$G(\omega) = \det G(\omega) \times \begin{pmatrix} (\omega_2 - \omega) + \dfrac{\pi\lambda^2}{2\rho_2(\sqrt{\omega} + i\rho_2)^2} & -\dfrac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} \\ -\dfrac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)} & (\omega_1 - \omega) + \dfrac{\pi\lambda^2}{2\rho_1(\sqrt{\omega} + i\rho_1)^2} \end{pmatrix}. \tag{44}$$

The determinant $\det G(\omega)$ is

$$(\det G(\omega))^{-1} = \left[\omega_1 - \omega + \frac{\pi\lambda^2}{2\rho_1(\sqrt{\omega} + i\rho_1)^2}\right]\left[\omega_2 - \omega + \frac{\pi\lambda^2}{2\rho_2(\sqrt{\omega} + i\rho_2)^2}\right] - \left(\frac{\pi\lambda^2}{(\rho_1 + \rho_2)(\sqrt{\omega} + i\rho_1)(\sqrt{\omega} + i\rho_2)}\right)^2. \tag{45}$$

Here we can formulate necessary conditions for the analyticity of the function $G_{kk'}^{-1}$ on the first sheet:

$$\begin{aligned} &1.\ \omega_i \rho_i^2 - \frac{\pi \lambda^2}{2\rho_i} > 0, \quad i = 1, 2 \\ &2.\ \left(\omega_1 \rho_1^2 - \frac{\pi \lambda^2}{2\rho_1}\right)\left(\omega_2 \rho_2^2 - \frac{\pi \lambda^2}{2\rho_2}\right) > \left(\frac{\pi \lambda^2}{\rho_1 + \rho_2}\right)^2. \end{aligned} \tag{46}$$

These conditions are definitely satisfied in the weak coupling regime, because ω_i, ρ_i, and λ are independent parameters and for any fixed ω_i and ρ_i, in the limit $\lambda \to 0$ (46) becomes

$$\begin{aligned} &1.\ \omega_i \rho_i^2 > 0, \quad i = 1, 2 \\ &2.\ \omega_i \omega_2 \rho_i^2 \rho_2^2 > 0, \end{aligned}$$

which is obviously true as ω_i and ρ_i are positive for any i.

In order to find out the analytic structure of $G(\omega)$, we analyze the poles of the determinant:

$$\begin{aligned} (\det G(\omega))^{-1} = &\left[(\omega_1 + x^2)\,(x + \rho_1)^2 - \frac{\pi \lambda^2}{2\rho_1}\right] \\ &\times \left[(\omega_2 + x^2)\,(x + \rho_2)^2 - \frac{\pi \lambda^2}{2\rho_2}\right] - \left(\frac{\pi \lambda^2}{(\rho_1 + \rho_2)}\right)^2 = 0, \end{aligned} \tag{47}$$

where we substitute $\sqrt{\omega} = ix$. This is an algebraic equation of 8th degree with real coefficients, so all the roots of this equation are either real or complex conjugated pairs. All roots are on the second Riemann sheet, and there can be $k(k = 0 \ldots 4)$ pairs of complex conjugated roots and $(8 - 2k)$ real roots corresponding to virtual states, i.e., negative energy states on the second sheet. For weak coupling $\lambda \to 0$, two pairs of complex conjugated roots z_j, z_j^* can be evaluated perturbatively as

$$\begin{aligned} z_j = \omega_j &+ \frac{\pi \lambda^2}{2\rho_j} \frac{(\sqrt{\omega_j} - i\rho_j)^2}{(\omega_j + \rho_j^2)^2} + \frac{\pi^2 \lambda^4}{(\sqrt{\omega_j} + i\rho_j)^2} \\ &\times \left(\frac{1}{(\omega_j - \omega_k)\,(\rho_1 + \rho_2)^2\,(\sqrt{\omega_j} + i\rho_k)^2} - \frac{1}{4\rho_j^2 \sqrt{\omega_j}(\sqrt{\omega_j} + i\rho_j)^3} \right) \\ &+ O(\lambda^6), \quad j = 1, 2, \quad k \neq j. \end{aligned} \tag{48}$$

For the weak coupling regime the expressions for the real and imaginary parts of z_j are

$$\tilde{\omega}_j = \mathrm{Re} z_j = \omega_j + \frac{\pi \lambda^2}{2\rho_j} \frac{\omega_j - \rho_j^2}{(\omega_j + \rho_j^2)^2} + O(\lambda^4), \quad j = 1, 2,$$

$$\gamma_j = -\mathrm{Im} z_j = \frac{\pi \lambda^2 \sqrt{\omega_j}}{(\omega_j + \rho_j^2)^2} + O(\lambda^4), \quad j = 1, 2.$$

Neglecting the integral term in the representation (39), we can write:

$$\begin{aligned} A(t) &\approx \sum_{k,k'=1,2} a_k a_{k'}^* \sum_{j=1,2} e^{-\gamma_j t} \, e^{-i\tilde{\omega}_j t} r_{kk'}^j \\ &= \sum_{k,k'=1,2} a_k a_{k'}^* \, e^{-i\frac{\tilde{\omega}_1+\tilde{\omega}_2}{2}t} \Big\{ \Big(r_{kk'}^1 \, e^{-\gamma_1 t} + r_{kk'}^2 \, e^{-\gamma_2 t} \Big) \cos \nu t \\ &\quad + i \left(r_{kk'}^1 \, e^{-\gamma_1 t} - r_{kk'}^2 \, e^{-\gamma_2 t} \right) \sin \nu t \Big\} \\ &= \sum_{j=1}^{2} |a_j|^2 \, e^{-iz_j t} - \lambda^2 \sum_{j=1}^{2} \left(\frac{i\pi |a_j^2| e^{-iz_j t}}{2\rho_j(\rho_j - i\sqrt{\omega_j})^3 \sqrt{\omega_j}} \right. \\ &\quad \left. + \frac{2\pi \ \mathrm{Re}(a_1 a_2^*) \, e^{-iz_j t}}{(\rho_1 + \rho_2)\,(\rho_j - i\sqrt{\omega_j})(\rho_l - i\sqrt{\omega_j})(\omega_j - \omega_l)} \right) + O(\lambda^4), \quad l \neq j, \end{aligned} \tag{49}$$

where

$$\nu = \frac{\tilde{\omega}_1 - \tilde{\omega}_2}{2}.$$

We would like to notice that both expressions (48) and (49) contain the term $1/(\omega_k - \omega_l)$ and, therefore, cannot be directly used in the case of degenerate levels in the initial Hamiltonian H_0. Also, the case of the continuous spectrum of H_0 requires a special consideration.

For the initial conditions $a_1 = 1$, $a_2 = 0$, the survival amplitude (49) does not have any oscillations. However, such oscillations appear in the next order λ^4 in expression (49). The survival probability $p(t)$ in the lowest order of λ^2 can be now expressed as

$$p(t) = |A(t)|^2 = \| a_1|^2 \, e^{-\gamma_1 t} + |a_2|^2 \, e^{-\gamma_2 t} \, e^{-2i\nu t}|^2. \tag{50}$$

We illustrate the possible behavior of the survival probability in Fig. 2. One can see that depending on the initial conditions, the decay can either mimic the behavior of the usual one level model (Antoniou *et al.*, 2001) or display considerable oscillations.

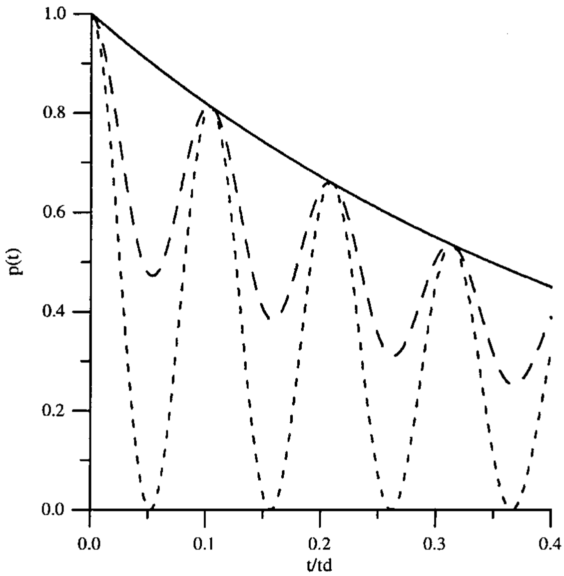

Fig. 2. The survival probability $p(t)$ for the two level model. The parameters are chosen to be $\gamma_1 = \gamma_2 = 10^{-3}$, $\omega_1 = 1.0$, $\omega_2 = 1.06$. The initial conditions are $a_1 = 0.5$, $a_2 = 0$ (the solid line), $a_1 = 0.5$, $a_2 = 0.2$ (the long-dashed line), $a_1 = 0.5$, $a_2 = 0.5$ (the short-dotted line). Time is in units of the decay time t_d.

5. *N*-LEVEL MODEL

In the weak coupling regime we can also analyze the N-level model with an arbitrary form factor $f_k(\omega)$. Using the representation (9), we find

$$(\det G(\omega))^{-1} = \prod_{k=1}^{N}(\omega_k - \omega) - \lambda^2 \sum_{k=1}^{N} I_{kk}(\omega) \prod_{m\neq k}^{N}(\omega_m - \omega) + O(\lambda^4), \qquad (51)$$

where

$$I_{kl}(\omega) = \int_0^\infty d\omega' \frac{f_k(\omega') f_l(\omega')}{\omega' - \omega - i0}.$$

The zeros of this expression give the position of resonances:

$$z_k = \omega_k - \lambda^2 I_{kk}(\omega_k) + O(\lambda^4) = \tilde{\omega}_k - i\gamma_k, \quad j = 1 \ldots N. \qquad (52)$$

In the first nontrivial order of the perturbation theory with respect to λ^2 we have:

$$\tilde{\omega}_k = \omega_k, \qquad \gamma_k = \pi\lambda^2 f_k^2(\omega_k).$$

The partial resolvent G can also be calculated:

$$G_{kk'}(\omega) = \left(\omega_k - \omega - \lambda^2 I_{kk'}\omega\right)^{-1} \delta_{kk'} + O(\lambda^2). \tag{53}$$

From this representation we obtain the expression for the residues (40):

$$r^j_{kk'} = -\delta_{kk'}\delta_{kj} + O(\lambda^2). \tag{54}$$

We derive the survival amplitude (39) in the first nonvanishing term of the perturbation expansion with respect to λ^2:

$$A(t) = \sum_{k=1}^{N} |a_k|^2 \, e^{-i\omega_k t} \, e^{-\pi\lambda^2 f_k^2(\omega_k)t}. \tag{55}$$

In the case of the N-level model, the behavior of the survival probability is much more complicated than in two level model. In order to illustrate this, we plot in Fig. 3 few examples of the survival probability corresponding to different initial

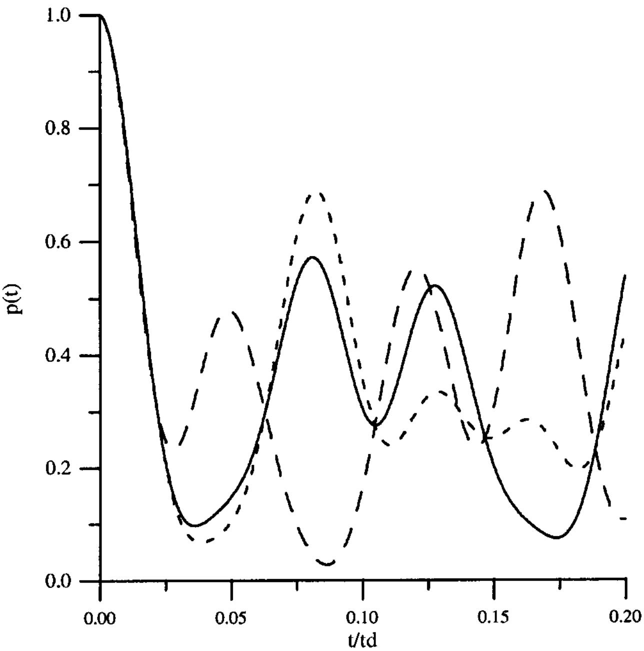

Fig. 3. The survival probability $p(t)$ for the three level model. The parameters are chosen to be $\gamma_1 = \gamma_2 = \gamma_3 = 10^{-3}$, $\omega_1 = 1$, the initial conditions are $a_1 = 0.3$, $a_2 = 0.5$, $a_2 = 0.3$. The energies are $\omega_2 = 1.04$, $\omega_3 = 1.15$ (the long-dashed line), $\omega_2 = 1.06$, $\omega_3 = 1.15$ (the solid line), $\omega_2 = 1.064$, $\omega_3 = 1.15$ (the short-dashed line). Time is in units of the decay time t_d.

conditions for the three level model with different parameters. In this case, the behavior is not necessarily "self-similar" even for the very slow decay. One can see that our curves reproduce fairly well the experimental results. Hence we can suggest here an explanation of the results (Cong *et al.*, 1996) which does not refer to the semiclassical description invoked in paper (Cong *et al.*, 1996). Namely, the initial laser impulse creates in the system *NaI* a wave packet which is a superposition of (many) excited states. Then each excited state decays independently while the common survival probability (55) exhibits a complicated behavior similar to one of Fig. 3 and Figs. 3, 4 in paper (Cong *et al.*, 1996).

In fact, the interference of many decaying states can drastically change the decay patterns. In this case, the decay is equally defined by both the parameters of the system (energies, widths) and the distribution of the initial wave packet a_k. Therefore, the decay profile may mimic different nonexponential functions. Such a behavior is illustrated in Fig. 4, where we plot the decay of the $(2N+1)$-level

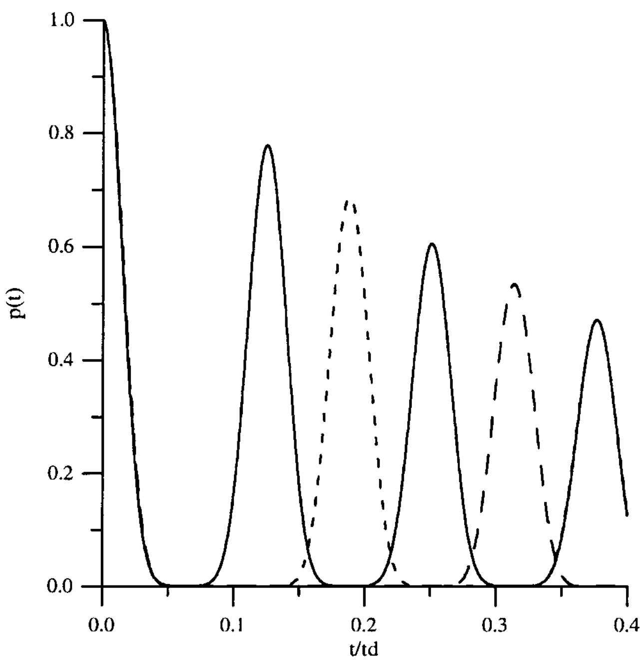

Fig. 4. The survival probability $p(t)$ for the N-level model. The parameters are chosen to be $\gamma_k = 10^{-3}$, $\omega_k = \omega_0 + k\Delta\omega/N$, $k = -N \dots N$, where $\omega_0 = 1$, $\Delta\omega = 0.1$. The initial conditions are $\tilde{a}_k = \exp(-(k/N)^2)$, $k = -N \dots N$. The results for $N = 2$ (the solid line), $N = 3$ (the short-dashed line), and $N = 5$ (the long-dashed line) are presented. Time is in units of the decay time t_d.

system:

$$\gamma_k = \gamma = \text{const}, \qquad \omega_k = \omega_0 + \frac{k}{N}\Delta\omega, \quad k = -N \dots N. \tag{56}$$

The initial distribution a_k is chosen to be the Gaussian one:

$$a_k = \frac{\tilde{a}_k}{\sum_k \tilde{a}_k^2}, \qquad \tilde{a}_k = \exp\left(-\left(\frac{k}{N}\right)^2\right), \quad k = -N \dots N. \tag{57}$$

From Fig. 4 we can see that the initial decay is almost independent of N. The duration of the initial decay is much shorter than the decay time $\tau_{\text{dec}} = 1/\gamma$. The number of repeated peaks decreases as N increases. Already for 10 levels, a rather small number for molecular systems, the second peak is very far from the region of the initial decay. While the values (56), (57) are chosen quite arbitararily and can only be used for illustrative purposes, we would like to notice that the excitation process in experiments similar to (Cong *et al.*, 1996) is usually well-defined and well-reproduced. Hence the initial wave packet may also be well-correlated.

6. CONCLUSIONS

Dissociation processes like the dissociation of NaI, which is a kind of tunneling/decay process, may be described by the simple quantum mechanical model of the interaction of the N-level discrete spectrum with the continuous spectrum. Already the model with two levels displays decaying oscillations of the survival probability in the “exponential” era, while one-level model exhibit the purely exponential decay. The amplitude of the oscillation is determined by the initial state, which is a superposition of two excited levels. The model with three levels may illustrate qualitatively the experimental curve of the NaI dissociation. In the N-level system, the decay is equally defined by both the parameters of the system and the distribution of the initial wave packet.

APPENDIX: TIME EVOLUTION IN TERMS OF GAMOW VECTORS

By analytic continuation to the second sheet, we obtain the extended distributions $G^d_{kl}(\omega \pm i0)$ and $1/[\omega - z_k]_+$ defined as functionals, which act on a suitable test function $h(\omega)$ as

$$\int_0^\infty d\omega h(\omega) G^d_{kl}(\omega \pm i0) \equiv \int_\Gamma h(\omega) G_{kl}(\omega \pm i0) \tag{A1}$$

$$= \int_0^\infty d\omega h(\omega) G_{kl}(\omega \pm i0) + 2\pi i \sum_j \int_{C_{z_j}} h(\omega) G_{kl}(\omega \pm i0),$$

$$= \int_0^\infty d\omega \frac{h(\omega)}{[\omega - z_j]_+} \equiv \int_\Gamma \frac{h(\omega)}{\omega - z_j} = \int_0^\infty d\omega \frac{h(\omega)}{\omega - z_j} + 2\pi i \int_{C_{z_j}} \frac{h(\omega)}{\omega - z_j} \quad \text{(A2)}$$

Using (A1) and (A2), we obtain from (12) the Gamow vectors (Antoniou and Prigogine, 1993; Bohm and Gadella, 1989; Petrosky *et al.*, 1991) in the form

$$\left|\phi_j^G\right\rangle = N_j \sum_{k,l=1}^{N} \lambda f_l(z_j) r_{kl}^j \left[|k\rangle - \int_0^\infty d\omega \frac{\lambda f_k(\omega)}{[\omega - z_j]_+} |\omega\rangle \right], \quad \text{(A3)}$$

$$\left\langle\tilde{\phi}_j^G\right| = N_j \sum_{k,l=1}^{N} \lambda f_l(z_j) r_{kl}^j \left[\langle k| - \int_0^\infty d\omega \frac{\lambda f_k(\omega)}{[\omega - z_j]_+} \langle\omega| \right], \quad \text{(A4)}$$

$$\left|\Psi_\omega^G\right\rangle = |\omega\rangle + \lambda \sum_{k,l=1}^{N} f_l(\omega) G_{kl}^d(\omega + i0) \left\{ \int_0^\infty \mathrm{d}\omega' \frac{\lambda f_k(\omega')}{\omega' - \omega - i0} |\omega'\rangle - |k\rangle \right\}, \quad \text{(A5)}$$

$$\left\langle\tilde{\Psi}_\omega^G\right| = \langle\omega|\lambda \sum_{k,l=1}^{N} f_l(\omega) G_{kl}(\omega - i0) \left\{ \int_0^\infty \mathrm{d}\omega' \frac{\lambda f_k(\omega')}{\omega' - \omega + i0} \langle\omega'| - \langle k| \right\} \quad \text{(A6)}$$

where $G_{kl}^d(\omega + i0)$ is a distribution with the kernel $G_{kl}(\omega + i0)$ and the contour of the integration Γ. We recall that r_{kl}^j is the residue of $G_{kl}(\omega + i0)$ at the pole z_j. The normalization constants N_k are

$$N_j^{-2} = \sum_{k,l,m,n=1}^{N} \lambda^2 f_l(z_j) f_n(z_j) r_{kl}^j r_{mn}^j \left[\delta_{km} + \int_0^\infty d\omega \frac{\lambda^2 f_k(\omega) f_m(\omega)}{[\omega - z_j]_+^2} \right]. \quad \text{(A7)}$$

The Gamow vectors (A3–A6) are left and right eigenfunctions of the extended Hamiltonian, which can be written as

$$H^+ = \sum_j z_j \left|\phi_j^G\right\rangle \left\langle\tilde{\phi}_j^G\right| + \int_0^\infty d\omega\, \omega \left|\Psi_\omega^G\right\rangle \left\langle\tilde{\Psi}_\omega^G\right|. \quad \text{(A8)}$$

The Gamow vectors from a biorthonormal set:

$$\left\langle\tilde{\phi}_j^G\middle|\phi_{j'}^G\right\rangle = \delta_{jj'}, \qquad \left\langle\tilde{\Psi}_\omega^G\middle|\Psi_{\omega'}^G\right\rangle = \delta(\omega - \omega'), \qquad \left\langle\tilde{\Psi}_\omega^G\middle|\phi_j^G\right\rangle = 0, \quad \text{(A9)}$$

which is complete. The completeness follows from the extension of (14):

$$I = \sum_{k=1}^{N} \left|\phi_k^G\right\rangle \left\langle\tilde{\phi}_k^G\right| + \int_0^\infty d\omega \left|\Psi_\omega^G\right\rangle \left\langle\tilde{\Psi}_\omega^G\right|. \quad \text{(A10)}$$

The time evolution of the vector $|k\rangle$ in the new extended representation is

$$|k\rangle_t = \sum_j e^{-iz_j t} \left|\phi_j^G\right\rangle \left\langle\tilde{\phi}_j^G \middle| k\right\rangle + \int_0^\infty d\omega\, e^{-i\omega t} \left|\Psi_\omega^G\right\rangle \left\langle \tilde{\Psi}_\omega^G \middle| k\right\rangle. \tag{A11}$$

Using (A3–A6), we express the transition amplitude

$$\begin{aligned} \langle k|k'\rangle_t &= \sum_j e^{-iz_j t} N_j^2 \sum_{l,l'=1}^{N} \lambda^2 f_l(\omega) f_{l'}(\omega) r_{kl}^j r_{k'l'}^j \\ &+ \sum_{l,l'=1}^{N} \int_0^\infty d\omega\, e^{-i\omega t} \lambda^2 f_l(\omega) f_{l'}(\omega) G_{kl}^d(\omega + i0) G_{k'l'}(\omega - i0). \end{aligned} \tag{A12}$$

The integral terms of (A12) can be rewritten in the form

$$\frac{1}{2\pi i} \int_0^\infty d\omega\, e^{-i\omega t} \left(G_{kk'}^d(\omega + i0) - G_{kk'}(\omega - i0) \right). \tag{A13}$$

Taking into account that $G_{kk'}^d(\omega + i0)$ implies integration along the contour Γ, which goes to the second Riemann sheet below all the singularities of $G_{kk'}(\omega + i0)$ (see Fig. A1), we obtain the transition amplitude in the form

$$\langle k|k\rangle_t = \sum_j e^{-iz_j t} N_j^2 \sum_{k,l,k',l'=1}^{N} \lambda^2 f_l(z_j) f_{l'}(z_j) r_{kl}^j r_{k'l'}^j + \frac{1}{2\pi i} \int_{C_1} \mathrm{d}\omega\, e^{-i\omega t} G_{kk'}, \tag{A14}$$

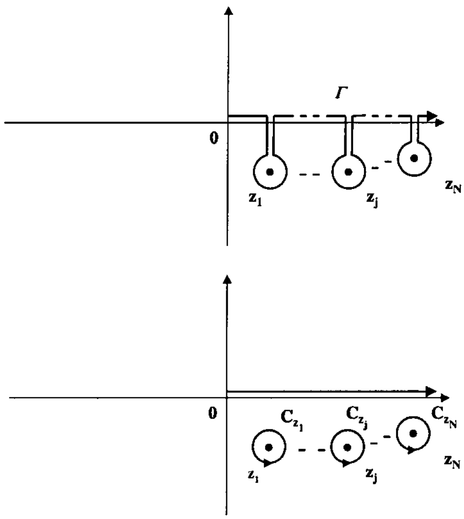

Fig. A1. The contours of integration Γ and C_{z_j}.

which must coincide with the result obtained using Friedrichs solution (39). In order to fulfill this requirement the following formula must hold:

$$N_j^2 \sum_{k,l,k',l'=1}^{N} \lambda^2 f_l(z_j) l_l'(z_j) r_{kl}^j r_{k'l'}^j = -r_{kk'}^j. \tag{A15}$$

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REFERENCES

Alicki, R. and Lendi, K. (1987). *Quantum Dynamical Semigroups and Applications* (Springer Lecture Notes in Physics, Vol. 286), Springer, Berlin.

Antoniou, I., Karpov, E., Pronko, G., and Yarevsky, E. (2001). Quantum Zeno and anti-zeno effects in the Friedrichs model. *Physical Review A* **63**, 062110.

Antoniou, I. and Prigogine, I. (1993). Intrinsic irreversibility and integrability of dynamics. *Physica A* **192**, 443.

Balzer, Chr., Huesmann, R., Neuhauser, W., and Toschek, P. E. (2000). The quantum Zeno effect-evolution of an atom impeded by measurement. *Optics Communications* **180**, 115.

Bailey, T. K. and Schieve. W. C. (1978). Complex Energy Eigenstates in Quantum Decay Models. *Nouvo Cimento A* **47**, 231.

Bohm, A. and Gadella, M. (1989). *Dirac Kets, Gamow Vectors and Gelfand Triplets*, (Springer Lecture Notes in Physics, Vol. 348), Springer, Berlin.

Cong, P., Roberts, G., Herek, J. L., Mohkatari, A., and Zewail, A. H. (1996). Femtosecond Real-Time Probing of Reactions. 18. Experimental and Theoretical Mapping of Trajectoraes and Potentials in the Nal Dissociation Reaction. *Journal of Physical Chemistry* **100**, 7832.

Davies, E. B. (1974). Dynamics of a multilevel Wigner-Wejsskopf atom. *Journal of Mathematical Physics* **15**, 2036.

Duerinckx, G. (1983). On the point spectrum of the N-points Friedrichs model. *Journal of Physics A* **16**, L289.

Exner, P. (1985). *Open Quantum Systems and Feynman Integrals*, Reidel, Dordrecht, The Netherlands.

Facchi, P., Nakazato, H., and Pascazio, S. (2001). From the Quantum zeno to the Inverse Quantum zeno effect. *Physical Review Letters* **86**, 2699.

Facchi, P. and Pascazio, S. (1999). Deviations from exponential law and Van Hove's "$\lambda^2 t$" limit. *Physica A* **271**, 133.

Felker, P. M. and Zewail, A. H. (1995). In: *Jet Spectroscopy and Molecular Dynamics*, M. Hollas and D. Philips, eds., Chapman and Hall, New York.

Fischer, M. C., Gutierrez-Medina, B., and Raizen, M. G. (2001). Observation of the Quantum zeno and Anti-Zone Effects in an Unstable System. *Physical Review Letters* **87**, 040402.

Friedrichs, K. (1948). On the periurbation of continuous spectra. *Communications in Pure and Applied Mathematics* **1**, 361.

Gaspard, P. and Burghardt, I. (eds.) (1997). Chemical reactions and their control on the femtosecond time scale. In *Advances in Chemical Physics, Vol. 101*, John Wiley & Sons, New York.

Hegerfeldt, G. C. and Plenio M. B. (1992). Macroscopic dark periods without a metastable state. *Physical Review A* **46**, 373.

Hegerfeldt, G. C. and Plenio, M. B. (1993). Coherence with incoherent light: A new type of quantum beats for a single atom. *Physical Review A* **47**, 2186.

Imre, D., Kinsey, J. L., Sinha, A., and Krenos, J. (1984). Chemical dynamics studied by emission spectroscopy of dissociating molecules. *Journal of Physical Chemistry* **88**, 3956.

Itano, W. M., Heinzen, D. J., Bollinger, J. J., and Wineland, D. J. (1990). Quantum zeno effect. *Physical Review A* **41**, 2295.

Khalfin, L. A. (1957). Contribution to the theory of decay of a quasi-stationary state. *Soviet Physics Doklady* **2**, 340.

Khalfin, L. A. (1958). Contribution to the decay theory of a quasi-stationary state. *Soviet Physics JETP* **6**, 1053.

Kofman, A. G. and Kurizki, G. (2000). Acceleration of quantum decay processes by frequent obsorvations. *Nature (London)* **405**, 546.

Kofman, A. G., Kurizki, G., and Sherman, B. (1994). Spontaneous and induced atomic decay in photonic band structures. *Journal of Modern Optics* **41**, 353.

Lendi, K. (1980). On quantum beats in polyatomic molecules. *Chemical Physics* **46**, 179.

Lienau, C. and Zewail, A. H. (1996). Solvation Ultrafast Dynamics of Reactions. 11. Dissociation and Caging Dynamics in the Gas-to-Liquid Transition Region. *Journal of Physical Chemistry* **100**, 18629.

Likhoded, A. and Pronko, G. (1997). Possible Origin of Extra States in Particle Physics. *International Journal of Theoretical Physics* **36**, 2335.

Namiki, M., Pascazio, S., and Nakazato, H. (1997). *Decoherence and Quantum Measurements*, World Scientific, Singapore.

Ordonez, G., Petrosky, T., and Prigogine, I. (2001). Quantum transitions and dressed unstable states. *Physical Review A* **63**, 052106.

Petrosky, T., Prigogine, I., and Tasaki, S. (1991). Quantum theory of non-integrable systems. *Physica A* **173**, 175.

Potter, E. D., Herek, J. L., Pedersen, S., Liu, Q., and Zewail, A. H. (1992). Femtosecond laser control of a chemical reaction. *Nature* **355**, 66.

Ruuskanen, P. V. (1970). Non-exponential decays in a Lee model with several unstable V-particles. *Nuclear Physics B* **22**, 253.

Stey, G. C. and Gibberd, R. W. (1972). Decay of quantum states in some exactly soluble models. *Physica* **60**, 1.

Zewail, A. H. (2000). Femtochemistry: Atomic-Scale Dynamics of the Chemical Bond. *Journal of Physical Chemistry A* **104**, 5660.